Eliciting discrete mixtures of piecewise linear additive preference models

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Résumé

Eliciter un modèle de préférences additif consiste à poser une suite de questions à un décideur pour identifier une fonction de valeur additive représentant ses préférences. Dans ce travail, nous considérons à un groupe de décideurs avec pour objectif d'éliciter une fonction de valeur pour représenter les préférences de chacun. Nous considérons le cas où on questionne deux décideurs, mais sans savoir quelle réponse correspond à quel décideur. Nous proposons une procédure d'élicitation identifiant les deux modèles de préférences lorsque les fonctions marginales représentées par des fonctions additives linéaires par morceaux.

Mots-clés

Modèle de préférence additif linéaire par morceaux, élicitation de modèles multiples, identification.

Abstract

Eliciting an additive preference model involves asking a decision maker a series of questions to identify an additive value function representing her preferences. In this work, we consider a group of decision-makers, and we aim to elicit a value function representing each decision-maker's preferences. We restrict to the case where two decision makers provide answers, but without knowing which answer corresponds to which decision maker. We propose an elicitation procedure that identifies the two preference models when the marginal functions are represented by piecewise linear additive functions.

Keywords

Additive piecewise linear preference model, multiple model elicitation, identification.

1 Introduction

The additive value model is standard to specify preferences over a set of alternatives evaluated on several criteria, see [9]. More precisely, given an alternative x defined by its evaluation on n criteria $x = (x_1, \ldots, x_n)$, where x_i is the evaluation of x on criterion i, the value of x is defined by $u(x) = \sum_i u_i(x_i)$, where u_i is the marginal value function on criterion i. In this paper, we consider that the marginals v_i are strictly increasing and piecewise linear, with an a priori defined linear decomposition. In the following, we will call any such preference model an UTA model. To elicit such an UTA model, standard queries correspond to matching questions involving two alternatives x and y whose evaluations differ on two criteria i and j only; three evaluations out of x_i, x_j, y_i, and y_j are given, and the respondent provides the unfixed value so that indifference holds.

We consider a framework in which we have to elicit simultaneously two UTA models. In this context, the answer to a matching query corresponds to two values for y_i, but without the ability to identify to which model each answer corresponds. In this paper, our purpose is to study this problem from an identifiability perspective. In other words, is it always possible to define a finite sequence of queries whose answers enables to identify the two piecewise linear additive value models ? The identifiability problem can also be formulated as a game in which a first player specifies the queries, and second player provides (thruthfull) answers; is there a finite winning strategy for the first player ? in another variant of the game (not considered in this paper), the second player can answer with an (adversarial) noise.

Although the problem is formulated as a purely formal one, it is worth noticing that such a problem may correspond to real-world problems :

1. A supermarket is willing to adapt the list of products to its customer base. The products selected should align with the client's preferences. Yet, not all customers necessarily have the same preferences, and it is standard to consider a market segmentation in which each segment represents a group of clients with homogeneous preferences.
2. A website about cinema requests user feedback on films. Users provide film scores from which pairwise comparisons can be derived. Users are not requested to have an account and are hence unidenti-

fiable. For marketing purposes, the website wants to identify, from the collected data, clusters of users, with each a specific preference model.

3. A group of oenologists is performing a blind tasting evaluation of various wines. Wines have been previously evaluated from a chemical point of view, and the concentration of several chemical components known to characterize the quality of the wines are measured. Each expert is required to attach to wines a number of "stars" (from ⋆ to ⋆⋆⋆⋆⋆). All experts in the panel obviously are not fully aligned in the way they assess wines. We would like to identify groups of experts with similar tastes and how each group accounts for wine characteristics in their preferences.

Such type of problem has already been tackled from a preference learning perspective [2], and the identifiability result proposed is of high importance to justify the search for efficient learning algorithms to infer multiple UTA models from preference statements.

Our primary aim is to provide a systematic elicitation procedure to identify two additive piecewise linear value functions from a finite sequence of matching queries. In this introduction, we provide an overall presentation of this procedure ; the following sections aim at providing a detailed and justified description of this elicitation procedure. Hence, the main result of the paper can be formulated as follows.

Theorem 1.1 *It is possible to elicit a mixture of two piecewise linear additive preference models using a finite number of queries.*

In our setting, each matching query involves two alternatives x and y that vary on two criteria i and j only, so that three values out of x_i, x_j, y_i, and y_j are fixed and the response corresponds to two evaluations a and a' for the unfixed component to obtain an indifference statement in each of the two value functions to be elicited. Given that marginal value functions are piecewise linear, it is convenient to define two queries with a specific geometry that we will use as "building blocks" of our elicitation procedure :

— Interval-constrained query : both evaluations on one criterion are selected within a chosen linear segment, and on the other criterion the fixed value, and both answers a and a' are in the same linear segment.

— Straddling query : both evaluations on one criterion are selected within a chosen linear segment, and on the other criterion the fixed value, and one answer a are in the same linear segment, but the answer a' is in the segment just above.

Interval-constrained queries make it possible to elicit the anonymized pair of slopes of marginal value functions on a specific interval, while straddling queries provide coupling constraints allowing to assign the slopes to the two value functions.

Using these building blocks, the procedure starts with the first two criteria, and proceeds with an initialization followed by an iteration :

— Initialization : the procedure starts with the two first criteria and postulates w.l.o.g. that the slope on criterion 1 is equal to one on the first segment. Then, an interval-constrained query identifies the slope of each model for the first segment on the second criterion.

— Iteration : given the pair of slopes of both value models on criterion 1 for segment ℓ, a set of straddling queries allows to identify the slopes of both models on criterion 1 for segment $\ell + 1$; the same applies to criterion 2.

Finally, in order to obtain the marginal value functions on all criteria, one can apply the same principle for all pairs of criteria $(1,i)$, $i \neq 1$.

The paper is organized as follows. After a review of related works in Section 2, Section 3 provides general notation. Section 4 presents the two query types which constitute "building blocks" for the identification procedure. Section 5 presents results justifying the iterative nature of the general procedure presented in Section 6.

2 Background and Related Work

The additive value (or utility) function (AVF) model is the most widely used and studied preference model. It may represent preference relations on sets of objects $x = (x_1, \ldots, x_n)$ described by n attributes or criteria. If attribute i takes values in a set $\mathcal{X}_i$ (criterion scale), let us consider a preference relation P on the Cartesian product $\mathcal{X} = \Pi_{i=1}^n \mathcal{X}_i$. Under well understood conditions on the preference P [9, 8, 13], there exist real valued functions $u_i : \mathcal{X}_i \to \mathbb{R}$ such that, for all $x, y \in \mathcal{X}$, we have

$$xPy \quad \text{iff} \quad \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i).$$

The marginal value (or utility) functions u_i are unique up to positive affine transformations. The main property required of P for admitting such a representation is an independence condition allowing for *ceteris paribus* reasoning. Specifically, one may compare objects without specifying the values of the attributes on which they agree. Such characterization results allow one to devise query strategies for eliciting the marginal value functions, hence describing the preferences on the whole Cartesian product by a numerical representation. Among the query strategies, the method of indifference judgments builds a "standard sequence" of points on a criterion scale, which are equally spaced (in terms of value). This is done by using a fixed interval on the scale of another criterion as a unit measure. Queries are designed as follows. Let q_i, p_i be two values on the scale $\mathcal{X}_i$ of criterion i, and q_j a value on the scale $\mathcal{X}_j$ of criterion j. The query is : What is the value $a_j \in \mathcal{X}_j$ such that the value difference between q_i and p_i on $\mathcal{X}_i$ is exactly compensated by the value difference between q_j and a_j on $\mathcal{X}_j$ (all other things being equal, i.e., ceteris paribus) ? The answer a_j is a first point in a standard sequence. The next point is the answer to the modified query obtained by substituting q_j by a_j in the initial query, and so on. The pairs (q_i, q_j), (p_i, a_j) be-

long to an indifference curve in the domain $\mathcal{X}_i \times \mathcal{X}_j$ (whence the name "indifference judgments").

Jacquet-Lagrèze and Siskos [6] developed a method (called UTA) to learn marginal value functions of an AVF that is compatible with known preferences expressed on a set of pairs of objects. In order to exploit linear programming they assume that the marginal value functions are piecewise linear. If the points $x_{i,l}, l = 0, 1, \ldots, L$ partition the scale $\mathcal{X}_i$ of criterion i into L subintervals $[x_{i,l}, x_{i,l+1}]$, and the marginal value function is piecewise linear with respect to this decomposition, then the latter is known as soon as the values $u_i(x_{i,l})$ at the intervals endpoints are determined. Alternatively, it suffices to determine the slopes of the linear segments of the marginal value function $\gamma_{i,l} = [u_i(x_{i,l}) - u_i(x_{i,l-1})]/(x_{i,l} - x_{i,l-1})$. This method has been subsequently widely developed and applied [7]. In general, there are several models (actually, an infinite number of models) compatible with the set of preference examples. Some papers [e.g., 5] deal with formulating robust conclusions by considering the set of all models compatible with the examples.

In this work, we consider the case in which two or more decision makers (DMs) reply to elicitation queries each using her own AVF model, with piecewise linear marginals. The answers to the queries are not tagged by the name of the DM. The question we tackle is whether it is possible to identify each DM's model without ambiguity, by asking queries of the type "indifference judgments". Model identifiability problems have been addressed in multicriteria decision analysis in the case of the Choquet integral [10] and the hierarchical Choquet integral [3].

The question of model identifiability can be approached from different angles. At its core is a parametric family of functions $(f_\omega)_{\omega \in \Omega}$ where Ω is the latent parameter set, and each f_ω is a function mapping queries in $\mathcal{Q}$ to answers in $\mathcal{A}$. Abstract identifiability can be cast as the existence of a query $q \in \mathcal{Q}$ such that two distinct parameters $\omega \neq \omega' \in \Omega$ yield distinct outcomes, i.e. $f_\omega(q) \neq f_{\omega'}(q)$. More operational definitions exist. For instance, various notions of combinatorial dimension have been put forward so as to upper bound the number of queries needed to distinguish concepts [1]. In turn, these bounds can be used to obtain statistical guarantees w.r.t. the estimation of those parameters from data, playing a key role in probably approximately correct machine learning (see e.g. [12, 11]). The problem we address here is not standard w.r.t. those approaches, because neither the queries nor the latent parameters are. Indifference queries do not have a binary outcome, as expected from concept learning theory. More importantly, our latent parameter space encompasses both the preference models of the DMs but also the identity of the DM associated to each answer (and thus grows with the number of queries). A more fruitful analogy comes from adversarial game-playing : consider two players, the Eliciter and the Adversary, and a fixed parameter space, such that ω completely specifies two UTA models. The Eliciter can either query the Adversary, or reveal two UTA models. The Adversary answers queries as he wishes. He immediately loses if there are no parameter ω compatible with all answers, and wins if the Eliciter reveals a parameter ω_E and he can provide an alternate $\omega' \neq \omega$ which is compatible with all answers. In effect, this allows the Adversary to cheat and modify the model behind the scene, as long as this behavior cannot be detected, and forces the Eliciter to reduce the set of all possible models to a singleton. Such approach can be found in e.g. [4]. Here, we show the Eliciter has indeed a winning strategy.

3 Notation

We consider a set $\mathcal{N} = \{1, ..., n\}$ of criteria. A criterion $i \in \mathcal{N}$ is defined on its scale, noted $\mathcal{X}_i \subset \mathbb{R}$. We note $x_{i,0}$ and $x_{i,L}$ the minimum and maximum value of the criterion and $\mathcal{X}_i$ can be decomposed into L equal sub-intervals of same size with the values $(x_{i,\ell})_{0 \leq \ell \leq L}$, $x_{i,\ell} = x_{i,0} + \ell \cdot \frac{x_{i,L} - x_{i,0}}{L}$.

A decision maker k whose value function is a UTA model can be defined by its values at the edges of the intervals, $(u_{k,i}(x_{i,\ell}))_{\ell \leq L, i \in \mathcal{N}}$. We write the slopes of this value function $\gamma_{k,i,\ell} = \frac{u_{k,i}(x_{i,\ell}) - u_{k,i}(x_{i,\ell-1})}{x_{i,\ell} - x_{i,\ell-1}}$ for $1 \leq \ell \leq L$. We consider the functions to be strictly increasing, meaning that $\gamma_{k,i,\ell} > 0$. The objective is to identify the different Decision Makers and their value function. For this purpose, it is possible to formulate a sequence of $\mathcal{T}$ queries.

We consider a group of heterogeneous decision makers, that can be queried at will. We define a query $\mathcal{Q}$ as questioning the decision makers concerning alternatives whose evaluations vary on two criteria only. More specifically, we formulate a query $Q^{(t)}$ as a specification of an ordered pair of criteria, $(i, j) \in \mathcal{N}^2$ and a triplet of values on these criteria : $(q_i^{(t)}, q_j^{(t)}, p_i^{(t)})$, with $(q_i^{(t)}, p_i^{(t)}) \in \mathcal{X}_i^2$ and $q_j^{(t)} \in \mathcal{X}_j$. This query triggers the collection of answers $\mathcal{A}^{(t)} = \{a_j^{\kappa,(t)}, a_j^{\kappa',(t)}\} \subset \mathcal{X}_j$ from the group of Decision Makers. We arbitrarily decide to note κ and κ' such that $a_j^{\kappa,(t)} \leq a_j^{\kappa',(t)}$. We also sometimes omit to specify the selected criteria, when it is already clear enough. The elements of $\mathcal{A}^{(t)}$ are such that either $\left[(q_i^{(t)}, q_j^{(t)}) \sim_\alpha (p_i^{(t)}, a_j^{\kappa,(t)}), (q_i^{(t)}, q_j^{(t)}) \sim_\beta (p_i^{(t)}, a_j^{\kappa',(t)})\right]$ or $\left[(q_i^{(t)}, q_j^{(t)}) \sim_\beta (p_i^{(t)}, a_j^{\kappa,(t)}), (q_i^{(t)}, q_j^{(t)}) \sim_\alpha (p_i^{(t)}, a_j^{\kappa',(t)})\right]$, with $\sim_\alpha$ (resp. $\sim_\beta$) denoting indifference for DM α (resp. β). In other words, the answers are non-identifiable, meaning that we don't known which decision maker they come from. A summary of the notation is in Table 1.

4 Queries geometry

Our elicitation algorithm consists in a sequence of different queries that are chosen to be in a specific criteria space. In this section, we show that such queries exist and describe a procedure to find one that follows the sought constraints. Several queries might be needed to obtain the final desired one. We consider that only the last query following the needed constraints is used in our elicitation sequence.

Description	Notation	
Criteria set	$\mathcal{N} = \{1, ..., n\}$	$i, j \in \mathcal{N}$
Criteria scale	$\mathcal{X}_i = [x_{i,0}, x_{i,L}]$	
Criteria Interval	$[x_{i,\ell}, x_{i,\ell+1}]$	$\ell \in \{0, ..., L\}$
Decision makers	α, β	
Value functions	$u^k(\cdot)$	$k \in \{\alpha, \beta\}$
Marginal value functions	$u_i^k(\cdot)$	$k \in \{\alpha, \beta\}, i \in \mathcal{N}$
Value function slopes	$\gamma_{i,\ell}^k$	$k \in \{\alpha, \beta\}, i \in \mathcal{N}, \ell \leq L$
Queries	$\mathcal{Q}^{(t)}$	
Query components	$q_i^{(t)}, p_j^{(t)}$	$i, j \in \mathcal{N}, q_i \in \mathcal{X}_i, p_j \in \mathcal{X}_j$
Answers	$a_i^{\kappa,(t)}, a_i^{\kappa',(t)}$	$i \in \mathcal{N}, a_i^{\kappa,(t)} \in \mathcal{X}_i$

TABLE 1 – Notation summary

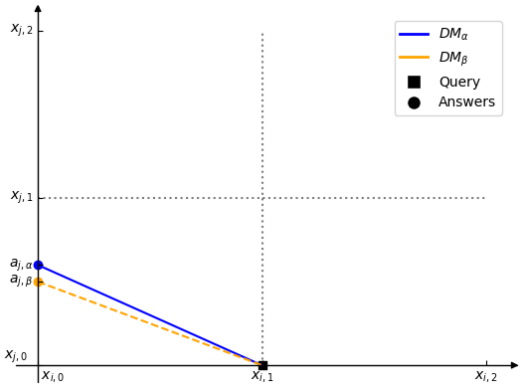


FIGURE 1 – Interval-constrained query, scenario (i). The colored lines represent the indifference curve of each DM.

4.1 Interval-constrained queries

The first type of query are queries constrained within intervals of our criteria cutout. Considering the ℓ_i-th and ℓ_j-th intervals of criteria $i, j \in \mathcal{N}$, we want to end up either with :

— $\mathcal{Q}_i^{(t)} = \langle (i,j), (q_i, q_j, p_i) \rangle$ and $\mathcal{A}_i^{(t)} = (a_{\kappa,j}, a_{\kappa',j})$ with $q_i^{(t)}, p_i^{(t)} \in [x_{i,\ell_i}, x_{i,\ell_i+1}]$ and $q_j^{(t)}, a_{\kappa,j}^{(t)}, a_{\kappa',j}^{(t)} \in [x_{j,\ell_j}, x_{j,\ell_j+1}]$

— $\mathcal{Q}_j^{(t)} = \langle (j,i), (q_j, q_i, p_j) \rangle$ and $\mathcal{A}_j^{(t)} = (a_{\kappa,i}, a_{\kappa',i})$ with $q_j^{(t)}, p_j^{(t)} \in [x_{j,\ell_j}, x_{j,\ell_j+1}]$ and $q_i^{(t)}, a_{\kappa,i}^{(t)}, a_{\kappa',i}^{(t)} \in [x_{i,\ell_i}, x_{i,\ell_i+1}]$

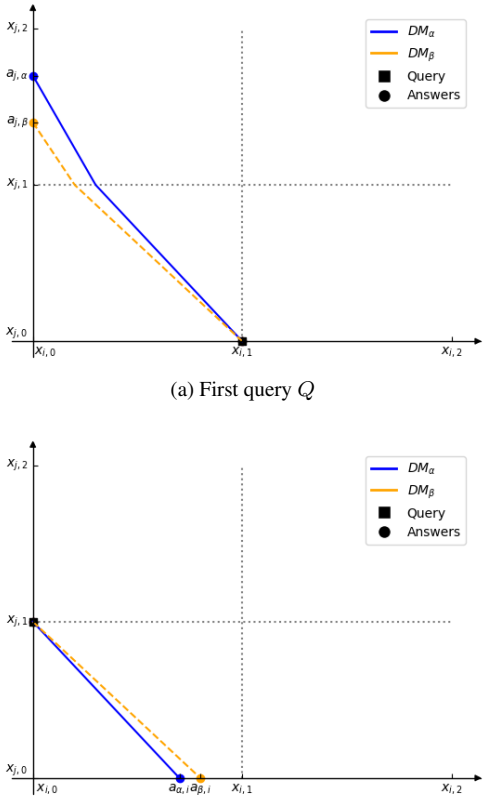
Lemma 4.1 *Given two criteria and an interval on each of those criteria, it is always possible to find an interval-constrained query using at most 3 queries.*

Procedure We formulate a first query from i to j : $Q = (x_{i,\ell_i+1}, x_{j,\ell_j}) \sim (x_{i,\ell_i}, ?)$. The DMs answer according to one scenario among three :

(i) The DMs answer with $a_{\kappa,j}, a_{\kappa',j} \in [x_{j,\ell_j}, x_{j,\ell_j+1}]$

(iii) Both answers returned are not in $[x_{j,\ell_j}, x_{j,\ell_j+1}]$, $a_{\kappa,j}, a_{\kappa',j} \geq x_{j,\ell_j+1}$

(iii) An answer $a_{\kappa,j} \in [x_{j,\ell_j}, x_{j,\ell_j+1}]$ and the second one $a_{\kappa',j} \geq x_{j,\ell_j+1}$



(a) First query Q

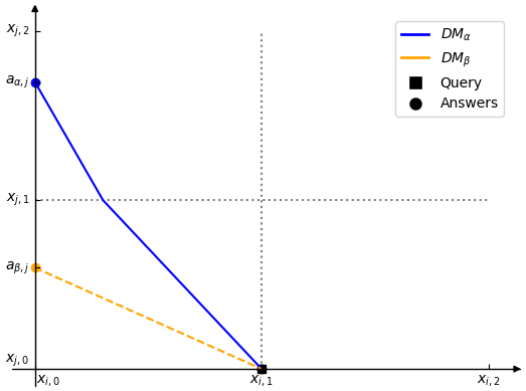
(b) Second query Q'

FIGURE 2 – Interval-constrained query, scenario (ii). The colored lines represent the indifference curve of each DM.

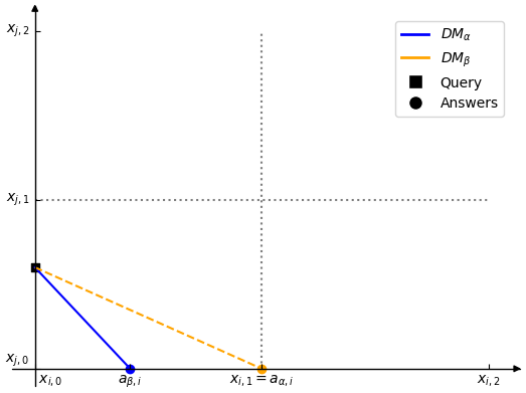
Scenario (i) In case (i), illustrated in Figure 1, both answers are within the right interval. We do not need to proceed to any further query and therefore $Q^{(t)} = Q$.

Scenario (ii) If both answers are such that $a_{\kappa,j}, a_{\kappa',j} \geq x_{j,\ell_j+1}$, it means that the utility difference yielded on criterion i, $u_{k,i}(x_{i,\ell_i+1}) - u_{k,i}(x_{i,\ell_i})$, is for both DM greater than the difference of utility on criterion j, $u_{k,j}(x_{j,\ell_j+1}) - u_{k,j}(x_{j,\ell_j})$. This case is illustrated on Figure 2a.

A second query from j to i, $Q' = (x_{j,\ell_j+1}, x_{i,\ell_i}) \sim (x_{j,\ell_j}, ?)$, makes sure that both of the new answers $a'_{\kappa,i}$ and



(a) First query Q



(b) Second query Q'

FIGURE 3 – Interval-constrained query, scenario (iii). The colored lines represent the indifference curve of each DM.

$a'_{\kappa',i}$ are within the interval $[x_{i,\ell_i}, x_{i,\ell_i+1}]$. This query is illustrated on Figure 2b. In this case, we can consider the query $Q^{(t)} = Q'$ or if we need the answers to be on the criterion j, we can query from i to j with $\mathcal{Q}^{(t)} = \left(a'_{\kappa,i}, x_{i,\ell_i}\right) \sim \left(x_{j\ell_j}, ?\right)$. $a'_{\kappa,i}$ being the smallest of two answers resulting from $\mathcal{Q}'$.

Scenario (iii) In this case, illustrated in Figure 3a, we specifically have one DM following scenario (i) and the other following scenario (ii). With the query from j to i $\mathcal{Q}' = (a'_{\kappa,j} x_{i,\ell_i}) \sim (x_{j,\ell_j}, ?)$, we make sure that both answers $a'_{\kappa,i}, a'_{\kappa',i} \in [x_{i,\ell_i}, x_{i,\ell_i+1}]$. The query is illustrated in Figure 3b. For this scenario, we can use $\mathcal{Q}^{(t)} = \mathcal{Q}'$ or if the answers need to be on criterion j, we can query from i to j with $\mathcal{Q}^{(t)} = \left(a'_{\kappa,i}, x_{i,\ell_i}\right) \sim \left(x_{j\ell_j}, ?\right)$.

4.2 Straddling queries

In addition to interval-constrained queries, we also need queries that involve two consecutive intervals on the "destination" criteria. Specifically, we want to build a query from i to j, $\mathcal{Q}^{(t)} : (q_i^{(t)}, q_j^{(t)}) \sim (p_i^{(t)}, ?)$ that is constrained within the interval $[x_{i,\ell_i}, x_{i,\ell_i+1}]$ and that is on either sides of x_{j,ℓ_j}. Differently said, we want $\mathcal{Q}^{(t)} = (q_i^{(t)}, q_j^{(t)}, p_i^{(t)})$ that triggers the answers $a_{\kappa,j}^{(t)}, a_{\kappa',j}^{(t)}$, such that :

- $q_i^{(t)}, p_i^{(t)} \in [x_{i,\ell_i}, x_{i,\ell_i+1}]$
- $q_j^{(t)} \in \left[x_{j,\ell_j-1}, x_{j,\ell_j}\right]$
- $a_{\kappa,j}^{(t)}, a_{\kappa',j}^{(t)} \in \left[x_{j,\ell_j}, x_{j,\ell_j+1}\right]$

Lemma 4.2 *Given a value $x_{j,\ell}$ on criterion j and an interval on another criterion i, it is always possible to find a straddling query over $x_{j,\ell}$ using the given interval on i with a finite number of intermediate queries.*

$q_i = \frac{1}{2}\cdot(x_{i,\ell_{i+1}} + x_{i,\ell_i})$, $q_j = \frac{1}{2}\cdot(x_{j,\ell_j} + x_{j,\ell_{j-1}})$ and $p_i = x_{i,\ell_i}$. From this point, three outcomes are possible, illustrated in Figure 4.

Scenario (i) Illustrated in Figure 4a, the scenario (i) corresponds to the desired result. The answers are in the right interval resulting in the desired query $\mathcal{Q}^{(t)} = \mathcal{Q}$.

Scenario (ii) Illustrated in Figure 4b, at least one of the answers is above x_{j,ℓ_j+1}. The next query will depend on the value of $\Delta a = a_{\kappa,j} - a_{\kappa',j}$.

- $\Delta a < x_{j,\ell_j+1} - x_{j,\ell_j}$:by formulating the query $\mathcal{Q}' = (q_i, q_j - a_{\kappa',j} + x_{k,\ell_j+1}) \sim (p_i, ?)$ we know that we will obtain answers such that : $a'_{\kappa,j} = x_{j,\ell_j+1}$and $x_{j,\ell_j+1} \geq a'_{\kappa',j} > x_{j,\ell_j}$. It means that we obtain what is needed, or scenario (i). The new query is illustrated in Figure 5a.
- $\Delta a > x_{j,\ell_j+1} - x_{j,\ell_j}$. At this point, the existence of a saddling query results from a continuity argument : by moving the reference point from (q_i, q_j), where both answers are above $x_{j,l_{j+1}}$, to (p_i, x_{j,l_j}), where both answers are below $x_{j,l_{j+1}}$, we can find a value where one answer is above $x_{j,l_{j+1}}$ and the other is below, because the answer is continuous w.r.t. the reference point. Finding an adequate reference point can be performed in a finite number of steps by binary search.

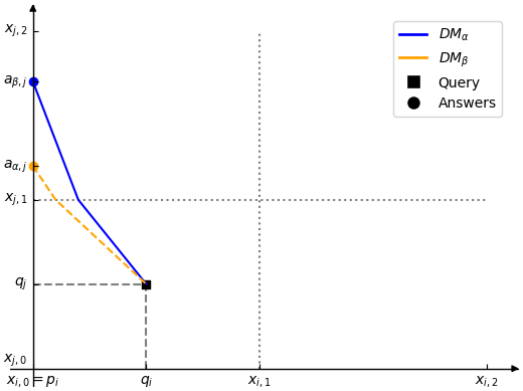
Scenario (iii) Illustrated in Figure 4c, at least one of the answers is below x_{i,ℓ_i}. The objective of the next query is to make sure that $a_{\kappa',j}, a_{\kappa',j} \geq x_{i,\ell_i}$. The idea is to choose $q'_j > q_j + x_{j,\ell_j} - a_{\kappa,j}$. Our proposition is to query $\mathcal{Q}' = (q_i, x_{j,\ell_j} + \frac{1}{2}\cdot(q_j - a_{\kappa,j})) \sim (p_i, ?)$. This query, illustrated in Figure 5b, can lead to either scenario (i) or (ii).
The scenario (iii) leading to (i) or (ii) makes sure to lean toward this convergence.

5 General identification properties

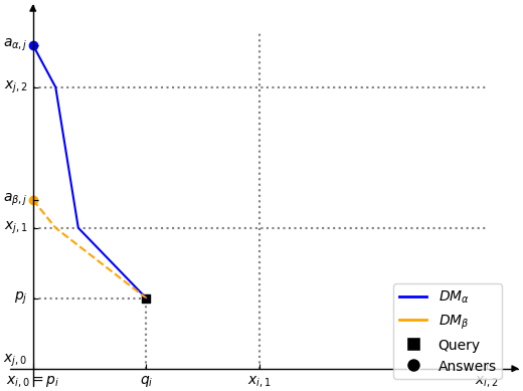
5.1 Identification with two interval-constrained query

Lemma 5.1 *It is possible to identify the slopes of both DMs value functions for a chosen interval ℓ on a given criterion i using two constrained queries involving two different intervals of another criterion j for which the marginal value functions are known.*

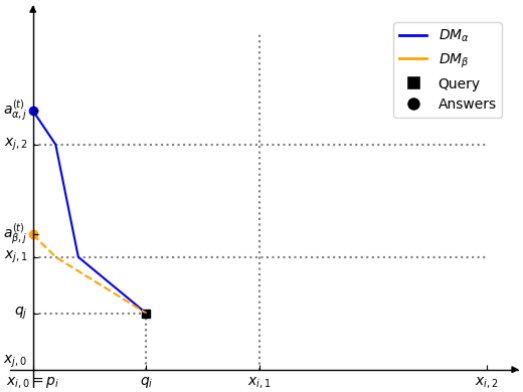
This Lemma means that once one marginal value function has been identified on at least two distinct intervals, it is possible to generalize the identifiability of any interval on another criterion. We propose a demonstration that leverages two constrained queries that helps formulate two distinct sets of equations.



(a) Possible outcome : scenario (i)

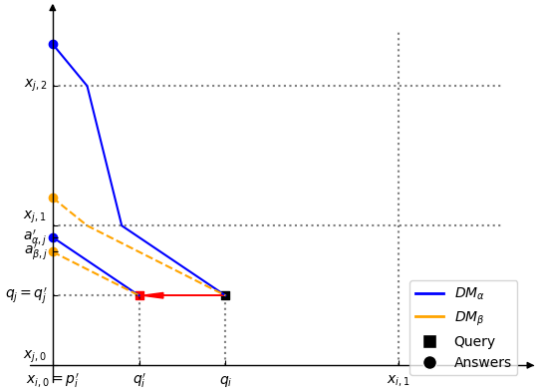


(b) Possible outcome : scenario (ii)

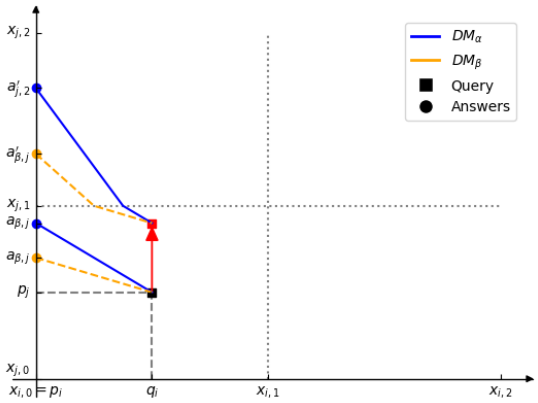


(c) Possible outcome : scenario (iii)

FIGURE 4 – Possible outcomes for a straddling query. The colored lines represent the indifference curve of each DM.



(a) Moving along criterion i



(b) Moving along criterion j

FIGURE 5 – Straddling query : two types of following query. The colored lines represent the indifference curve of each DM.

Proof : Let's consider an interval $[x_{i,\ell}, x_{i,\ell+1}]$ on a criterion i, for which the slopes $\gamma_{\alpha,i,\ell}, \gamma_{\beta,i,\ell}$ are to be elicited. We demonstrate that using two intervals on another criterion $j \neq i$, it is possible to identify the values of the slopes. We note these interval $[x_{j,0}, x_{j,1}]$ and $[x_{j,1}, x_{j,2}]$ for the sake of simplicity while keeping in mind that the demonstration works in the general case.

We obtain two different Equations involving the desired slopes, from which we can obtain their values. The first Equation is obtained with the query $\mathcal{Q}^{(t)}$ involving the interval $[x_{j,0}, x_{j,1}]$ and the second one is obtained through the query $\mathcal{Q}^{(t+1)}$ with the interval $[x_{j,1}, x_{j,2}]$.

We formulate the first constrained query $\mathcal{Q}^{(t)} = (q_i^{(t)}, q_j^{(t)}) \sim (p_i^{(t)}, ?)$ with $q_i^{(t)}, p_i^{(t)} \in [x_{i,\ell}, x_{i,\ell+1}]$ and $q_j^{(t)}, a_{\kappa,j}^{(t)}, a_{\kappa',j}^{(t)} \in [x_{j,0}, x_{j,1}]$. It means that :

$$
\begin{cases} (q_i^{(t)}, q_j^{(t)}) \sim_\kappa (p_i^{(t)}, a_{\kappa,j}^{(t)}) \\ (q_i^{(t)}, q_j^{(t)}) \sim_{\kappa'} (p_i^{(t)}, a_{\kappa',j}^{(t)}) \end{cases}
$$

$$
\Leftrightarrow \begin{cases} u_\kappa((q_i^{(t)}, q_j^{(t)})) = u_\kappa((p_i^{(t)}, a_{\kappa,j}^{(t)})) \\ u_{\kappa'}((q_i^{(t)}, q_j^{(t)})) = u_{\kappa'}((p_i^{(t)}, a_{\kappa',j}^{(t)})) \end{cases}
$$

$$
\Leftrightarrow \begin{cases} u_{\kappa,i}(x_{i,\ell-1}) + \gamma_{\kappa,i,\ell} \cdot (q_i^{(t)} - x_{i,\ell-1}) + \gamma_{\kappa,j,1} \cdot q_j^{(t)} \\ = u_{\kappa,i}(x_{i,\ell-1}) + \gamma_{\kappa,i,\ell} \cdot (p_i^{(t)} - x_{i,\ell-1}) + \gamma_{\kappa,j,1} \cdot a_{\kappa,j}^{(t)} \\ \\ u_{\kappa',i}(x_{i,\ell-1}) + \gamma_{\kappa',i,\ell} \cdot (q_i^{(t)} - x_{i,\ell-1}) + \gamma_{\kappa',j,1} \cdot q_j^{(t)} \\ = u_{\kappa',i}(x_{i,\ell-1}) + \gamma_{\kappa',i,\ell} \cdot (p_i^{(t)} - x_{i,\ell-1}) + \gamma_{\kappa',j,1} \cdot a_{\kappa,'j}^{(t)} \end{cases}
$$

$$
\Leftrightarrow \begin{cases} \gamma_{\kappa,i,\ell} = \dfrac{a_{\kappa,j}^{(t)} - q_j^{(t)}}{q_i^{(t)} - p_i^{(t)}} \cdot \gamma_{\kappa,j,1} \\ \gamma_{\kappa',i,\ell} = \dfrac{a_{\kappa',j}^{(t)} - q_j^{(t)}}{q_i^{(t)} - p_i^{(t)}} \cdot \gamma_{\kappa',j,1} \end{cases}
$$

We introduce the binary variable $k^{(t)}$ in order to resolve the ambiguity, with $a_{\alpha,j}^{(t)} = k^{(t)} \cdot a_{\kappa,j}^{(t)} + (1 - k^{(t)}) \cdot a_{\kappa',j}^{(t)}$ and $a_\beta = (1 - k^{(t)}) \cdot a_{\kappa,j}^{(t)} + k^{(t)} \cdot a_{\kappa',j}^{(t)}$. It leads to :

$$
\begin{cases} \gamma_{\alpha,i,\ell} = \left(k^{(t)} \cdot \Lambda_\kappa^{(t)} + (1 - k^{(t)}) \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\alpha,j,1} \\ \gamma_{\beta,i,\ell} = \left((1 - k^{(t)}) \cdot \Lambda_\kappa^{(t)} + k^{(t)} \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\beta,j,1} \end{cases} \tag{1}
$$

with $\Lambda_\kappa^{(t)} = \frac{a_{\kappa,j}^{(t)} - q_j^{(t)}}{q_i^{(t)} - p_i^{(t)}}$ and $\Lambda_{\kappa'}^{(t)} = \frac{a_{\kappa',j}^{(t)} - q_j^{(t)}}{q_i^{(t)} - p_i^{(t)}}$

We formulate the constrained query $\mathcal{Q}^{(t+1)} = (q_i^{(t+1)}, q_j^{(t+1)}) \sim (p_i^{(t+1)}, ?)$ with $q_i^{(t+1)}, p_i^{(t+1)} \in [x_{i,\ell}, x_{i,\ell+1}]$ and $q_j^{(t+1)}, a_{\kappa,j}^{(t+1)}, a_{\kappa',j}^{(t+1)} \in [x_{j,1}, x_{j,2}]$. Similarly, we obtain that :

$$
\begin{cases} (q_{i+1}^{(t+1)}, q_j^{(t+1)}) \sim_\kappa (p_i^{(t+1)}, a_{\kappa,j}^{(t+1)}) \\ (q_i^{(t+1)}, q_j^{(t+1)}) \sim_{\kappa'} (p_i^{(t+1)}, a_{\kappa',j}^{(t+1)}) \end{cases}
$$

$$
\Leftrightarrow \begin{cases} \gamma_{\kappa,i,\ell} = \dfrac{a_{\kappa,j}^{(t+1)} - q_j^{(t+1)}}{q_i^{(t+1)} - p_i^{(t+1)}} \cdot \gamma_{\kappa,j,2} \\ \gamma_{\kappa',i,\ell} = \dfrac{a_{\kappa',j}^{(t+1)} - q_j^{(t+1)}}{q_i^{(t+1)} - p_i^{(t+1)}} \cdot \gamma_{\kappa',j,2} \end{cases}
$$

In a similar fashion as in the previous query, we introduce the binary variable $k^{(t+1)}$ so that $\alpha = k^{(t+1)} \cdot \kappa + (1 - k^{(t+1)}) \cdot \kappa'$ and $\beta = (1 - k^{(t+1)}) \cdot \kappa + k^{(t+1)} \cdot \kappa'$. It leads to :

$$
\begin{cases} \gamma_{\alpha,i,\ell} = \left(k^{(t+1)} \cdot \Lambda_\kappa^{(t+1)} + (1 - k^{(t+1)}) \Lambda_{\kappa'}^{(t+1)}\right) \gamma_{\alpha,j,2} \\ \gamma_{\beta,i,\ell} = \left((1 - k^{(t+1)}) \Lambda_\kappa^{(t+1)} + k^{(t+1)} \cdot \Lambda_{\kappa'}^{(t+1)}\right) \gamma_{\beta,j,2} \end{cases} \tag{2}
$$

with $\Lambda_\kappa^{(t+1)} = \frac{a_{\kappa,j}^{(t+1)} - q_j^{(t+1)}}{q_i^{(t+1)} - p_i^{(t+1)}}$ and $\Lambda_{\kappa'}^{(t+1)} = \frac{a_{\kappa',j}^{(t+1)} - q_j^{(t+1)}}{q_i^{(t+1)} - p_i^{(t+1)}}$

Equating the right hand side of the first (respectively the second) equation in (1) and (2) yields a linear system of two equations with variables $k^{(t)}, k^{(t+1)}$. We show in Appendix A that these two equations are not linearly correlated unless : either $a_{\kappa,j}^{(t)} = a_{\kappa',j}^{(t)}$ or $a_{\kappa,j}^{(t+1)} = a_{\kappa',j}^{(t+1)}$ or $\frac{\gamma_{\beta,j,1}}{\gamma_{\alpha,j,1}} = \frac{\gamma_{\beta,j,2}}{\gamma_{\alpha,j,2}}$. In the first case, answers of both DMs are the same, we do not need any disjonction and can directly end up with desired values. In the second case, we argue that with a careful initialization of the value functions, presented in 6.1 , we have identified the value functions on two intervals such that those ratios cannot be the same. Therefore, we know two intervals that can be used. In the case where the two equations are not correlated, we can deduce the values of $k^{(t)}$ and $k^{(t+1)}$ resulting in obtaining the values for both desired slopes . Otherwise it means that at least one set of answers have been similar and the disjunction is not needed anymore. It is possible to directly deduce the values of $\gamma_{\alpha,i,\ell}, \gamma_{\beta,i,\ell}$ from the equations of this answer.

5.2 Slope identification with an interval-constrained and a straddling query

Lemma 5.2 *It is possible to identify the slopes of both DMs on any interval of criterion i using a straddling query with a neighboring interval on criterion i as well as any interval on another criterion j, for which the marginal value functions have been elicited.*

This Lemma means that once the marginal value functions have been identified on an interval for two different criteria, we will be able to extend the identifiability to the neighboring intervals of both criteria. Then, step by step, it will be possible to generalize the identifiability to all the intervals on both criteria.

We propose a demonstration that leverages one constrained query accompanied by a straddling query.

Proof : Let's consider an interval $[x_{i,\ell}, x_{i,\ell+1}]$ on a criterion i, for which the slopes $\gamma_{\alpha,i,\ell+1}, \gamma_{\beta,i,\ell+1}$ are to be elicited. We demonstrate that using the previous interval on criterion i, $[x_{i,\ell-1}, x_{i,\ell}]$ as well as any interval on another criterion $j \neq i$, it is possible to identify the values of the slopes. We note this interval $[x_{j,0}, x_{j,1}]$ for the sake of simplicity while keeping in mind that the demonstration works with any other interval.

We obtain two different Equations involving the desired slopes, from which we can derive their values. The first Equation is obtained with the query $\mathcal{Q}^{(t)}$ and the second one is obtained through the query $\mathcal{Q}^{(t+1)}$.

First, we formulate the interval-constrained query $\mathcal{Q}^{(t)} = (q_j^{(t)}, q_i^{(t)}) \sim (p_j^{(t)}, ?)$ with $q_j^{(t)}, p_j^{(t)} \in [x_{j,0}, x_{j,1}]$ and $q_i^{(t)} \in [x_{i,\ell}, x_{i,\ell+1}]$. The straddling query yields two answers, $a_{\kappa,i}^{(t)}, a_{\kappa',i}^{(t)} \in [x_{i,\ell}, x_{i,\ell+1}]$.

The same way as before, we obtain :

$$\begin{cases} (q_j^{(t)}, q_i^{(t)}) \sim_\kappa (p_j^{(t)}, a_{\kappa,j}^{(t)}) \\ (q_j^{(t)}, q_i^{(t)}) \sim_{\kappa'} (p_j^{(t)}, a_{\kappa',i}^{(t)}) \end{cases}$$
$$\Leftrightarrow \begin{cases} \gamma_{\kappa,i,\ell+1} = \dfrac{p_j^{(t)} - q_j^{(t)}}{q_i^{(t)} - a_{\kappa,i}^{(t)}} \cdot \gamma_{\kappa,j,1} \\ \gamma_{\kappa',i,\ell+1} = \dfrac{p_j^{(t)} - q_i^{(t)}}{q_i^{(t)} - a_{\kappa',i}^{(t)}} \cdot \gamma_{\kappa',j,1} \end{cases}$$

Once again, we model ambiguity with $k^{(t)} \in \{0,1\}$ such that $a_{\alpha,i}^{(t)} = k^{(t)} \cdot a_{\kappa,i}^{(t)} + (1-k^{(t)}) \cdot a_{\kappa',i}^{(t)}$ and $a_{\beta,i}^{(t)} = (1-k^{(t)}) \cdot a_{\kappa,i}^{(t)} + k^{(t)} \cdot a_{\kappa',i}^{(t)}$.

$$\begin{cases} \gamma_{\alpha,i,\ell+1} = k^{(t)} \cdot \left(\Lambda_\kappa^{(t)} - \Lambda_{\kappa'}^{(t)}\right) + \Lambda_{\kappa'}^{(t)} \\ \gamma_{\beta,i,\ell+1} = k^{(t)} \cdot \left(\Lambda_{\kappa'}^{(t)} - \Lambda_\kappa^{(t)}\right) + \Lambda_\kappa^{(t)} \end{cases} \tag{3}$$

with $\Lambda_\kappa^{(t)} = \frac{p_j^{(t)} - q_j^{(t)}}{q_i^{(t)} - a_{\kappa,i}^{(t)}}$ and $\Lambda_{\kappa'}^{(t)} = \frac{p_j^{(t)} - q_j^{(t)}}{q_i^{(t)} - a_{\kappa',i}^{(t)}}$

At this point we have deduced two slopes values for the $[x_{i,\ell}, x_{i,\ell+1}]$ interval, without being able to associate them to the right Decision Maker, and therefore need an additional query. The possible value functions are illustrated in Figure 6.

Then, it is possible to formulate the second query $\mathcal{Q}^{(t+1)} = (q_j^{(t+1)}, q_i^{(t+1)}) \sim (p_j^{(t+1)}, ?)$ with $q_j^{(t+1)}, p_j^{(t+1)} \in [x_{j,0}, x_{j,1}]^2$ and $q_i^{(t+1)} \in [x_{i,\ell-1}, x_{i,\ell}]\, a_{\kappa,i}^{(t+1)}, a_{\kappa',2}^{(t+1)} \in$

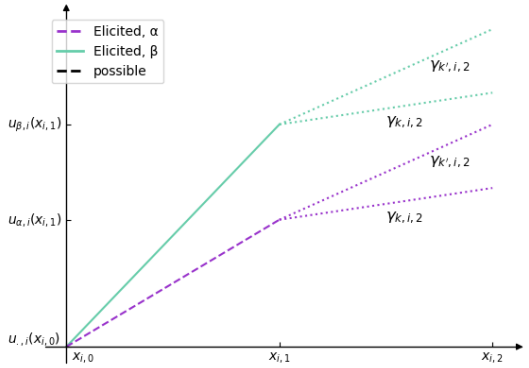


FIGURE 6 – Possible value functions after $\mathcal{Q}^{(t)}$

$[x_{i,\ell}, x_{i,\ell+1}]$. We find that :

$$\begin{cases} (q_j^{(t+1)}, q_i^{(t+1)}) \sim_\kappa (p_j^{(t+1)}, a_{\kappa,i}^{(t+1)}) \\ (q_j^{(t+1)}, q_i^{(t+1)}) \sim_{\kappa'} (p_j^{(t+1)}, a_{\kappa',i}^{(t+1)}) \end{cases}$$
$$\Leftrightarrow \begin{cases} \gamma_{\kappa,j,1} \cdot q_k^{(t+1)} + u_{\kappa,i}(x_{i,\ell-1}) + \gamma_{\kappa,i,\ell} \cdot (q_i^{(t+1)} - x_{i,\ell-1}) \\ = \gamma_{\kappa,j,1} \cdot p_j^{(t+1)} + u_{\kappa,i}(x_{i,\ell}) + \gamma_{\kappa,i,l+1} \cdot (a_{\kappa,i}^{(t+1)} - x_{i,\ell}) \\ \gamma_{\kappa',j,1} \cdot q_k^{(t+1)} + u_{\kappa',i}(x_{i,\ell-1}) + \gamma_{\kappa',i,\ell} \cdot (q_i^{(t+1)} - x_{i,\ell-1}) \\ = \gamma_{\kappa',j,1} \cdot p_j^{(t+1)} + u_{\kappa',i}(x_{i,\ell}) + \gamma_{\kappa',i,l+1} \cdot (a_{\kappa',i}^{(t+1)} - x_{i,\ell}) \end{cases}$$
$$\Leftrightarrow \begin{cases} \gamma_{\kappa,i,\ell+1} = \gamma_{\kappa,j,1} \cdot \dfrac{q_j^{(t+1)} - p_j^{(t+1)}}{a_{\kappa',i}^{(t+1)} - x_{i,\ell}} + \gamma_{\kappa,i,\ell} \cdot \dfrac{q_i^{(t+1)} - x_{i,\ell}}{a_{\kappa,i}^{(t+1)} - x_{i,\ell}} \\ \gamma_{\kappa',i,\ell+1} = \gamma_{\kappa',j,1} \cdot \dfrac{q_j^{(t+1)} - p_j^{(t+1)}}{a_{\kappa',i}^{(t+1)} - x_{i,\ell}} + \gamma_{\kappa',i,\ell} \cdot \dfrac{q_i^{(t+1)} - x_{i,\ell}}{a_{\kappa',i}^{(t+1)} - x_{i,\ell}} \end{cases}$$

We remind that $u_{\kappa,i}(x_{i,\ell}) = \sum_{l=1}^{l=\ell} \gamma_{\kappa,i,l} \cdot (x_{i,l} - x_{i,l-1})$ and thus $u_{\kappa,i}(x_{i,\ell}) - u_{\kappa,i}(x_{i,\ell-1}) = \gamma_{\kappa,i,\ell} \cdot (x_{i,\ell} - x_{i,\ell-1})$.
We introduce $k^{(t+1)} \in \{0,1\}$ such that $\gamma_{\alpha,i,\ell+1} = k^{(t+1)} \cdot \gamma_{\kappa,i,\ell+1} + (1-k^{(t+1)}) \cdot \gamma_{\kappa',i,\ell+1}$. It results that :

$$\begin{cases} \gamma_{\alpha,i,\ell+1} = \left[k^{(t+1)}\Theta_\kappa^{(t+1)} + (1-k^{(t+1)})\Theta_{\kappa'}^{(t+1)}\right]\gamma_{\alpha,j,1} \\ \qquad + \left[k^{(t+1)}\Lambda_\kappa^{(t+1)} + (1-k^{(t+1)})\Lambda_\kappa^{(t+1)}\right]\gamma_{\alpha,i,\ell} \\ \gamma_{\beta,i,\ell+1} = \left[(1-k^{(t+1)})\Theta_\kappa^{(t+1)} + k^{(t+1)}\Theta_{\kappa'}^{(t+1)}\right]\gamma_{\alpha,j,1} \\ \qquad + \left[(1-k^{(t+1)})\Lambda_\kappa^{(t+1)} + k^{(t+1)}\Lambda_{\kappa'}^{(t+1)}\right]\gamma_{\beta,i,\ell} \end{cases} \tag{4}$$

with :

$$\begin{cases} \Theta_\kappa^{(t+1)} = \dfrac{q_j^{(t+1)} - p_j^{(t+1)}}{a_{\kappa,i}^{(t+1)} - x_{i,\ell}}, \Theta_{\kappa'}^{(t+1)} = \dfrac{q_j^{(t+1)} - p_j^{(t+1)}}{a_{\kappa',i}^{(t+1)} - x_{i,\ell}} \\ \Lambda_\kappa^{(t+1)} = \dfrac{q_i^{(t+1)} - x_{i,\ell}}{a_{\kappa,i}^{(t+1)} - x_{i,\ell}}, \Lambda_{\kappa'}^{(t+1)} = \dfrac{q_i^{(t+1)} - x_{i,\ell}}{a_{\kappa',i}^{(t+1)} - x_{i,\ell}} \end{cases}$$

Equating the right hand side of the first (respectively the second) equation in 3 and 4 yields a linear system of two equations with variables $k^{(t)}, k^{(t+1)}$.

We show in Appendix B that these two Equations are not linearly correlated unless : Either $a^{(t)}_{\kappa,j} = a^{(t)}_{\kappa',j}$ or $a^{(t+1)}_{\kappa,j} = a^{(t+1)}_{\kappa',j}$ or $\frac{\gamma_{\beta,j,1}}{\gamma_{\alpha,j,1}} = \frac{\gamma_{\beta,i,\ell}}{\gamma_{\alpha,i,\ell}}$. In the first case, answers of both DMs are the same we do not need any disjunction and can directly end up with desired values. In the second case, we argue that with a careful initialization of the value functions, presented in 6.1 , we have identified the value functions on two intervals such that those ratios cannot be the same. Therefore, we know two intervals that can be used.

6 General Elicitation Procedure

We propose a general elicitation procedure that can be used in order to identify a set of two decision makers in our described setup. Let's first consider the first two criteria, $\mathcal{N}_c = \{1, 2\}$.

6.1 Initialization

During the initialization step, we make sure that the value functions have been identified on a minimal number of intervals.

Normalization of $\gamma_{.,1,1}$ We decide to normalize all the value functions as follows : $\forall k \in \{\alpha, \beta\}, \gamma_{k,1,1} = 1$, meaning that

$$\gamma_{\alpha,1,1} = \gamma_{\beta,1,1} = 1 \tag{5}$$

This can be done without loss of generality, as the marginal value functions elicited are unique up to a positive affine transformation, and it is always possible to normalize a posteriori the elicited value functions such that they take their value in the unit interval.

Distinction of the two DMs, α and β The first query will be used to differentiate α aims at eliciting the function slopes on the first interval of the second criterion. We formulate the query $\mathcal{Q}^{(1)} = (q^{(1)}_1, q^{(1)}_2) \sim (p^{(1)}_1, ?)$ with $q^{(1)}_1 = x_{1,0}$, $p^{(1)}_1 \in [x_{1,0}, x_{1,1}]$ and $q^{(2)}_1 = x_{2,1}$. We obtain the answers $a^{(1)}_{\kappa,2}$ and $a^{(1)}_{\kappa',2}$. We arbitrarily decide to attribute the answer $a^{(1)}_{\kappa,2}$ to the DM α and $a^{(1)}_{\kappa',2}$ to β, leading to :

$$\begin{cases} u_\alpha((q^{(1)}_1, q^{(1)}_2) = u_\alpha((p^{(1)}_1, a^{(1)}_{\kappa,2})) \\ u_\beta((q^{(1)}_1, q^{(1)}_2) = u_\beta((p^{(1)}_1, a^{(1)}_{\kappa',2})) \end{cases}$$
$$\Leftrightarrow \begin{cases} u_{\alpha,1}(q^{(1)}_1) + u_{\alpha,2}(q^{(1)}_2) = u_{\alpha,1}(p^{(1)}_1) + u_{\alpha,2}(a^{(1)}_{\kappa,2}) \\ u_{\beta,1}(q^{(1)}_1) + u_{\beta,2}(q^{(1)}_2) = u_{\beta,1}(p^{(1)}_1) + u_{\beta,2}(a^{(1)}_{\kappa',2}) \end{cases}$$
$$\Leftrightarrow \begin{cases} \gamma_{\alpha,1,1} \cdot q^{(1)}_1 + \gamma_{\alpha,2,1} \cdot q^{(1)}_2 = \gamma_{\alpha,1,1} \cdot p^{(1)}_1 + \gamma_{\alpha,2,1} \cdot a^{(1)}_{\kappa,2} \\ \gamma_{\beta,1,1} \cdot q^{(1)}_1 + \gamma_{\beta,2,1} \cdot q^{(1)}_2 = \gamma_{\beta,1,1} \cdot p^{(1)}_1 + \gamma_{\beta,2,1} \cdot a^{(1)}_{\kappa',2} \end{cases}$$

With the normalized values described in the normalization Section, we obtain :

$$\begin{cases} \gamma_{\alpha,2,1} = \dfrac{p^{(1)}_1 - q^{(1)}_1}{q^{(1)}_2 - a^{(1)}_{\kappa,2}} \\ \gamma_{\beta,2,1} = \dfrac{p^{(1)}_1 - q^{(1)}_1}{q^{(1)}_2 - a^{(1)}_{\kappa',2}} \end{cases} \tag{6}$$

Handling the special case where $\frac{\gamma_{\beta,2,1}}{\gamma_{\alpha,2,1}} = 1$. In the Lemmas 5.2, 5.1, we argue that we need two intervals where the ration of the slopes value is different. The initialization makes sure that these two intervals exist, unless $\frac{\gamma_{\beta,2,1}}{\gamma_{\alpha,2,1}} = 1$. However, it means that the value functions from the two DMs are not distinguished yet since they are still the same. Therfore we can re-iterate the second part of the initialization until the Equation 6 actually leads to two different slope values and two intervals with different ratios, letting us use the two Lemmas.

6.2 Iteration

Using the intervals $[x_{1,0}, x_{1,1}]$ and $[x_{2,0}, x_{2,1}]$, it is now possible to formulate straddling queries, following lemma 5.2, in order to identify the value functions on intervals $[x_{1,1}, x_{1,2}]$ and $[x_{2,1}, x_{2,2}]$. Then working step by step, using a straddling query on intervals $[x_{i,\ell-1}, x_{i,\ell}]$ and $[x_{j,0}, x_{j,1}]$with $1 \leq \ell \leq L-1$ and $i, j \in \{1, 2\}$, it is possible to identify the value functions on interval $[x_{i,\ell}, x_{i,\ell+1}]$. With such a method, it is possible to generalize the identifiability to the whole scale of the first two criteria.

Now, for any criterion $i \in \mathcal{N}, i > 2$, it is possible to identify the value functions on any interval $[x_{i,\ell}, x_{i,\ell+1}]$. Indeed, using lemma 5.1, with the criterion 1 and the intervals $[x_{1,0}, x_{1,1}]$ and $[x_{1,1}, x_{1,2}]$, lets obtain the desired results.

Conclusion

In this paper, we study the identifiability of two additive piecewise linear value functions from matching queries that provide two answers (those of the two additive models), but without knowing to which model each answer corresponds. We provide a systematic elicitation procedure that makes it possible to specify the two additive piecewise linear models from a finite sequence of matching queries, hence proving identifiability.

This work leaves open questions that should be studied in further research. First, our work is limited to additive value functions for which the marginals are piecewise linear ; it would be interesting to study this identifiability problem when marginals can be any monotonically increasing function. A second extension relates to the fact that our work accounts for two decision makers only ; extending interval-constrained and straddling queries to more than two decision makers seems straightforward, but defining an elicitation procedure to identify n additive models ($n > 2$) would be of great interest. Finally, the positive results concerning the identifiability of such model could help to analyze the behavior of statistical learning algorithms.

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A Appendix A : Condition of unicity of the system resulting from Equations 1, 2

We have two expressions, for the values of $\gamma_{\alpha,i,\ell}, \gamma_{\beta,i,\ell}$ resulting from Equations1 and 2, precisely :

$$\begin{cases} \gamma_{\alpha,i,\ell} = \left(k^{(t)} \cdot \Lambda_{\kappa}^{(t)} + (1-k^{(t)}) \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\alpha,j,1} \\ \gamma_{\beta,i,\ell} = \left((1-k^{(t)}) \cdot \Lambda_{\kappa}^{(t)} + k^{(t)} \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\beta,j,1} \end{cases}$$

$$\begin{cases} \gamma_{\alpha,i,\ell} = \left(k^{(t+1)} \cdot \Lambda_{\kappa}^{(t+1)} + (1-k^{(t+1)})\Lambda_{\kappa'}^{(t+1)}\right) \gamma_{\alpha,j,2} \\ \gamma_{\beta,i,\ell} = \left((1-k^{(t+1)})\Lambda_{\kappa}^{(t+1)} + k^{(t+1)} \cdot \Lambda_{\kappa'}^{(t+1)}\right) \gamma_{\beta,j,2} \end{cases}$$

Those values being the same, it results that :

$$\begin{cases} \left(k^{(t)} \cdot \Lambda_{\kappa}^{(t)} + (1-k^{(t)}) \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\alpha,j,1} = \left(k^{(t+1)} \cdot \Lambda_{\kappa}^{(t+1)} + (1-k^{(t+1)})\Lambda_{\kappa'}^{(t+1)}\right) \gamma_{\alpha,j,2} \\ \left((1-k^{(t)}) \cdot \Lambda_{\kappa}^{(t)} + k^{(t)} \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\beta,j,1} = \left((1-k^{(t+1)})\Lambda_{\kappa}^{(t+1)} + k^{(t+1)} \cdot \Lambda_{\kappa'}^{(t+1)}\right) \gamma_{\beta,j,2} \end{cases}$$
$$\Leftrightarrow \begin{cases} \left(k^{(t)} \cdot (\Lambda_{\kappa}^{(t)} - \Lambda_{\kappa'}^{(t)}) + \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\alpha,j,1} = \left(k^{(t+1)} \cdot (\Lambda_{\kappa}^{(t+1)} - \Lambda_{\kappa'}^{(t+1)}) + \Lambda_{\kappa'}^{(t+1)}\right) \gamma_{\alpha,j,2} \\ \left(k^{(t)} \cdot (\Lambda_{\kappa'}^{(t)} - \Lambda_{\kappa}^{(t)}) + \Lambda_{\kappa}^{(t)}\right) \cdot \gamma_{\beta,j,1} = \left(k^{(t+1)} \cdot (\Lambda_{\kappa'}^{(t+1)} - \Lambda_{\kappa}^{(t+1)}) + \Lambda_{\kappa}^{(t+1)}\right) \gamma_{\beta,j,2} \end{cases}$$

If the equations are linearly correlated, it means that the coefficients in $k^{(t)}, k^{(t+1)}$ are correlated between the equations. Differently said, it would mean that $z \in \mathbb{R}$ such that :

$$\begin{cases} (\Lambda_{\kappa}^{(t)} - \Lambda_{\kappa'}^{(t)})\gamma_{\alpha,j,1} = z \cdot (\Lambda_{\kappa'}^{(t)} - \Lambda_{\kappa}^{(t)})\gamma_{\beta,j,1} \\ (\Lambda_{\kappa'}^{(t+1)} - \Lambda_{\kappa}^{(t+1)})\gamma_{\alpha,j,2} = z \cdot (\Lambda_{\kappa}^{(t+1)} - \Lambda_{\kappa'}^{(t+1)})\gamma_{\beta,j,2} \end{cases}$$
$$\Leftrightarrow \begin{cases} \gamma_{\alpha,j,1} = -z \cdot \gamma_{\beta,j,1} \\ \gamma_{\alpha,j,2} = -z \cdot \gamma_{\beta,j,2} \end{cases}$$

The second system holds if the Λ values are different at t and $t+1$. Which is the case if the answers of the two DMs are different. It results that the solution of the system is unique unless :

- the answers from the query t or $t+1$ are the same for both DMs
- the slopes ratios of each DM are the same on the two intervals on criterion j, $\frac{\gamma_{\alpha,j,2}}{\gamma_{\beta,j,2}} = \frac{\gamma_{\alpha,j,1}}{\gamma_{\beta,j,1}}$

B Appendix B : Condition of Unicity for the solution of the system resulting from Equations 3 and 4

We have two expressions, for the values of $\gamma_{\alpha,i,\ell+1}, \gamma_{\beta,i,\ell+1}$ resulting from Equations 3, 4. In this appendix we find conditions of unicity for the solution of this system. Precisely, we have :

$$\begin{cases} \gamma_{\alpha,i,\ell+1} = \left[k^{(t)} \cdot \Lambda_{\kappa}^{(t)} + (1-k^{(t)}) \cdot \Lambda_{\kappa'}^{(t)}\right] \gamma_{\alpha,j,1} \\ \gamma_{\beta,i,\ell+1} = \left[(1-k^{(t)}) \cdot \Lambda_{\kappa}^{(t)} + k^{(t)} \cdot \Lambda_{\kappa'}^{(t)}\right] \gamma_{\beta,j,1} \end{cases}$$

$$\begin{cases} \gamma_{\alpha,i,\ell+1} = \left[k^{(t+1)}\Theta_{\kappa}^{(t+1)} + (1-k^{(t+1)})\Theta_{\kappa'}^{(t+1)}\right] \gamma_{\alpha,j,1} + \left[k^{(t+1)}\Lambda_{\kappa}^{(t+1)} + (1-k^{(t+1)})\Lambda_{\kappa}^{(t+1)}\right] \gamma_{\alpha,i,\ell} \\ \gamma_{\beta,i,\ell+1} = \left[(1-k^{(t+1)})\Theta_{\kappa}^{(t+1)} + k^{(t+1)}\Theta_{\kappa'}^{(t+1)}\right] \gamma_{\alpha,j,1} + \left[(1-k^{(t+1)})\Lambda_{\kappa}^{(t+1)} + k^{(t+1)}\Lambda_{\kappa'}^{(t+1)}\right] \gamma_{\beta,i,\ell} \end{cases}$$

Those values being the same, it results that :

$$\begin{cases} \left(k^{(t)} \cdot \Lambda_{\kappa}^{(t)} + (1-k^{(t)}) \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\alpha,j,1} = \left[k^{(t+1)}\Theta_{\kappa}^{(t+1)} + (1-k^{(t+1)})\Theta_{\kappa'}^{(t+1)}\right] \gamma_{\alpha,j,1} + \left[k^{(t+1)}\Lambda_{\kappa}^{(t+1)} + (1-k^{(t+1)})\Lambda_{\kappa}^{(t+1)}\right] \gamma_{\alpha,i,\ell} \\ \left((1-k^{(t)}) \cdot \Lambda_{\kappa}^{(t)} + k^{(t)} \cdot \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\beta,j,1} = \left[(1-k^{(t+1)})\Theta_{\kappa}^{(t+1)} + k^{(t+1)}\Theta_{\kappa'}^{(t+1)}\right] \gamma_{\alpha,j,1} + \left[(1-k^{(t+1)})\Lambda_{\kappa}^{(t+1)} + k^{(t+1)}\Lambda_{\kappa'}^{(t+1)}\right] \gamma_{\beta,i,\ell} \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(k^{(t)} \cdot (\Lambda_{\kappa}^{(t)} - \Lambda_{\kappa'}^{(t)}) + \Lambda_{\kappa'}^{(t)}\right) \cdot \gamma_{\alpha,j,1} = (k^{(t+1)}(\Theta_{\kappa}^{(t+1)} - \Theta_{\kappa'}^{(t+1)}) + \Theta_{\kappa'}^{(t+1)})\gamma_{\alpha,j,1} + (k^{(t+1)}(\Lambda_{\kappa}^{(t+1)} - \Lambda_{\kappa}^{(t+1)}) + \Lambda_{\kappa}^{(t+1)})\gamma_{\alpha,i,\ell} \\ \left(k^{(t)} \cdot (\Lambda_{\kappa'}^{(t)} - \Lambda_{\kappa}^{(t)}) + \Lambda_{\kappa}^{(t)}\right) \cdot \gamma_{\beta,j,1} = (k^{(t+1)}(\Theta_{\kappa'}^{(t+1)} - \Theta_{\kappa}^{(t+1)}) + \Theta_{\kappa}^{(t+1)})\gamma_{\beta,j,1} + (k^{(t+1)}(\Lambda_{\kappa'}^{(t+1)} - \Lambda_{\kappa'}^{(t+1)}) + \Lambda_{\kappa}^{(t+1)})\gamma_{\beta,i,\ell} \end{cases}$$

If the equations are linearly correlated, it means that the coefficients in $k^{(t)}, k^{(t+1)}$ are correlated between the equations. Differently said, it would mean that :

$$\begin{cases} (\Lambda_{\kappa}^{(t)} - \Lambda_{\kappa'}^{(t)})\gamma_{\alpha,j,1} = z \cdot (\Lambda_{\kappa'}^{(t)} - \Lambda_{\kappa}^{(t)})\gamma_{\beta,j,1} \\ (\Theta_{\kappa}^{(t+1)} - \Theta_{\kappa'}^{(t+1)})\gamma_{\alpha,j,1} + (\Lambda_{\kappa}^{(t+1)} - \Lambda_{\kappa'}^{(t+1)})\gamma_{,i,\ell} = z \cdot \left[(\Theta_{\kappa'}^{(t+1)} - \Theta_{\kappa}^{(t+1)})\gamma_{\beta,j,1} + (\Lambda_{\kappa'}^{(t+1)} - \Lambda_{\kappa}^{(t+1)})\gamma_{\beta,i,\ell}\right] \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma_{\alpha,j,1} = -z \cdot \gamma_{\beta,j,1} \\ \gamma_{\alpha,j,2} = z \cdot \gamma_{\beta,j,} \end{cases}$$

If $a_{\kappa,i}^{(t)} \neq a_{\kappa',i}^{(t)}$ and $a_{\kappa,i}^{(t+1)} \neq a_{\kappa',i}^{(t+1)}$, $\Lambda_{\kappa}^{(t)} - \Lambda_{\kappa'}^{(t)} \neq 0, \Lambda_{\kappa}^{(t+1)} - \Lambda_{\kappa'}^{(t+1)} \neq 0, \Theta_{\kappa}^{(t+1)} - \Theta_{\kappa'}^{(t+1)} \neq 0$. We can write :

$$\begin{cases} \gamma_{\alpha,j,1} = -z \cdot \gamma_{\beta,j,1} \\ (\Theta_{\kappa}^{(t+1)} - \Theta_{\kappa'}^{(t+1)})\gamma_{\alpha,j,1} + (\Lambda_{\kappa}^{(t+1)} - \Lambda_{\kappa'}^{(t+1)})\gamma_{,i,\ell} = -\dfrac{\gamma_{\alpha,j,1}}{\gamma_{\beta,j,1}} \cdot \left[(\Theta_{\kappa'}^{(t+1)} - \Theta_{\kappa}^{(t+1)})\gamma_{\beta,j,1} + (\Lambda_{\kappa'}^{(t+1)} - \Lambda_{\kappa}^{(t+1)})\gamma_{\beta,i,\ell}\right] \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma_{\alpha,j,1} = -z \cdot \gamma_{\beta,j,1} \\ \gamma_{\alpha,i,\ell} = \dfrac{\gamma_{\alpha,j,1}}{\gamma_{\beta,j,1}} \cdot \gamma_{\beta,i,\ell} \end{cases}$$

Similarly to A, we find that the solution of the system is unique unless :

— the answers from the query t or $t+1$ are the same for both DMs

— the slopes ratios of each DM are the same on the two intervals on criterion j, $\frac{\gamma_{\alpha,i,\ell}}{\gamma_{\beta,i,\ell}} = \frac{\gamma_{\alpha,j,1}}{\gamma_{\beta,j,1}}$